YY'.

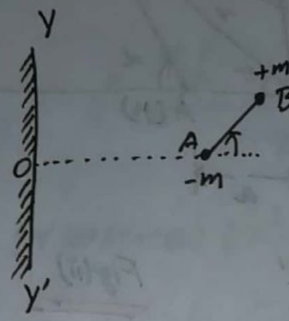


Fig. (iii)

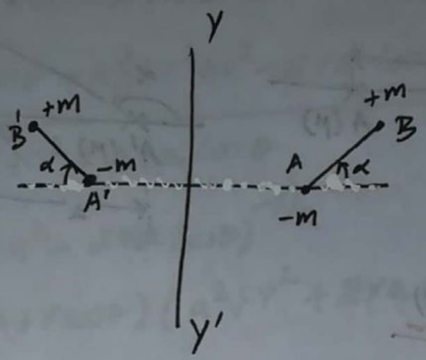


fig (iv)

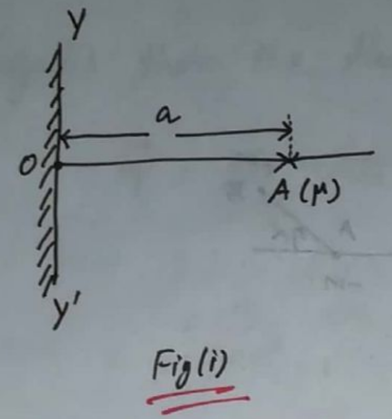
Consider a pair of Sources $-m$ at A, m at B close together and on one side of the rigid plane YY' in fig (iii).

The image system is $-m$ at A', m at B', where A', B' are the respective optic images of the point A, B in the plane YY' in fig (iv).

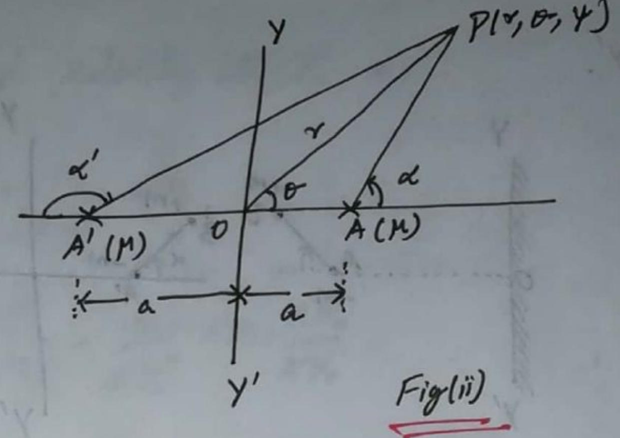
In the limit it follows, then, the image of a doublet in an infinite rigid plane is an equal doublet symmetrically disposed with respect to the plane.

Example

1) A three-dimensional doublet of strength μ whose axis is in the direction $\overline{Ox}$ is distance 'a' from the rigid plane $x=0$ which is the sole boundary of liquid of density ρ, infinite in extend. Find the pressure at a point on the boundary distant r from the doublet given that the pressure at infinite is p_∞. Show that the pressure on the plane is least at a distance $a\sqrt{5}/2$ from doublet.



Fig(i)



Fig(ii)

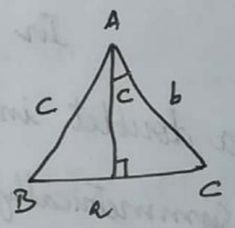
Fig(i) Shows the Physical model.

YY' being a rigid boundary.

The appropriate image system is shown in Fig (ii) in which P having spherical polar coordinates (r, θ, ψ) is the considered field point in the region $0 \leq \theta \leq \pi/2$.

Thus the velocity potential at P is

$$\phi = \frac{\mu \cos\alpha}{AP^2} + \frac{\mu \cos\alpha'}{A'P^2} \longrightarrow (1) \quad \left(\because \phi = \frac{\mu \cos\alpha}{r^2}\right)$$



In ΔOAP,

$$AP^2 = r^2 + a^2 - 2ra\cos\theta$$

$$a = r\cos\theta + AP\cos(\pi - \alpha)$$

$$= r\cos\theta - AP\cos\alpha$$

$$\cos\alpha = \frac{r\cos\theta - a}{AP} = \frac{r\cos\theta - a}{(r^2 + a^2 - 2ra\cos\theta)^{1/2}}$$

$$\cos\alpha = (r\cos\theta - a)(r^2 + a^2 - 2ra\cos\theta)^{-1/2}$$

In $\Delta A'OP$,

$$A'P^2 = a^2 + r^2 - 2ra\cos(\pi - \theta)$$

$$= a^2 + r^2 + 2ra\cos\theta$$

$$a = r\cos(\pi - \theta) + A'P\cos(\pi - \alpha')$$

$$= -r\cos\theta - A'P\cos\alpha'$$

$$\cos\alpha' = \frac{-r\cos\theta - a}{A'P} = \frac{-r\cos\theta - a}{(r^2 + a^2 + 2ra\cos\theta)^{1/2}}$$

$$= -(r\cos\theta + a)(r^2 + a^2 + 2ra\cos\theta)^{-1/2}$$

$$(1) \Rightarrow \quad \phi = \frac{\mu(r\cos\theta - a)(r^2+a^2-2ra\cos\theta)^{-1/2}}{r^2+a^2-2ra\cos\theta}$$

$$+ \frac{\mu[-(r\cos\theta+a)](r^2+a^2-2ra\cos\theta)^{-1/2}}{r^2+a^2+2ra\cos\theta}$$

$$\phi = \mu\Big[(r\cos\theta-a)(r^2+a^2-2ra\cos\theta)^{-3/2} - (a+r\cos\theta)(a^2+r^2+2ra\cos\theta)^{-3/2}\Big]$$

Hence $q_r = \dfrac{-\partial\phi}{\partial r}$

$$= -\mu\Big\{(\cos\theta)(r^2+a^2-2ra\cos\theta)^{-3/2} + (r\cos\theta-a)(-\tfrac{3}{2})(r^2+a^2-2ra\cos\theta)^{-5/2}(2r-2a\cos\theta)$$

$$- \Big[(\cos\theta)(a^2+r^2+2ra\cos\theta)^{-3/2} + (a+r\cos\theta)(-\tfrac{3}{2})(r^2+a^2+2ra\cos\theta)^{-5/2}(2r+2a\cos\theta)\Big]\Big\}$$

$$= -\mu\Big\{(\cos\theta)(r^2+a^2-2ra\cos\theta)^{-3/2} - (\tfrac{3}{\cancel{2}})(r\cos\theta-a)(\cancel{2})(r-a\cos\theta)(r^2+a^2-2ra\cos\theta)^{-5/2} - \cos\theta(a^2+r^2+2ra\cos\theta)^{-3/2}$$

$$- \tfrac{3}{\cancel{2}}(\cancel{2})(a+r\cos\theta)(r^2+a^2+2ra\cos\theta)(r+a\cos\theta)\Big\}$$

$$= -\mu\Big\{\cos\theta(r^2+a^2-2ra\cos\theta)^{-3/2} - 3(r\cos\theta-a)(r-a\cos\theta)(r^2+a^2-2ra\cos\theta)^{-5/2} - \cos\theta(r^2+a^2-2ra\cos\theta)^{-3/2}$$

$$+ 3(r+a\cos\theta)(r\cos\theta+a)(r^2+a^2+2ra\cos\theta)^{-5/2}\Big\}$$

$$q_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\mu}{r}\Big\{(-r\sin\theta)(r^2+a^2-2ra\cos\theta)^{-3/2} + (r\cos\theta-a)(-\tfrac{3}{\cancel{2}})(r^2+a^2-2ra\cos\theta)^{-5/2}\cdot(\cancel{2}ra\sin\theta)$$

$$- \Big[(-r\sin\theta)(a^2+r^2+2ra\cos\theta)^{-3/2} + (a+r\cos\theta)(-\tfrac{3}{\cancel{2}})(a^2+r^2+2ra\cos\theta)^{-5/2}(-\cancel{2}ra\sin\theta)\Big]\Big\}$$

$$= -\frac{\mu}{r}\Big\{-r\sin\theta(r^2+a^2-2ar\cos\theta)^{-3/2} - 3ra\sin\theta(r\cos\theta-a)(r^2+a^2-2ra\cos\theta)^{-5/2} + r\sin\theta(r^2+a^2+2ra\cos\theta)^{-3/2}$$

$$- 3ra\sin\theta(r\cos\theta+a)(r^2+a^2+2ra\cos\theta)^{-5/2}\Big\}$$

$$q_\psi = -\frac{1}{r\sin\theta}\cdot\frac{\partial\phi}{\partial\psi} = 0$$

when $\theta = \pi/2$

$$q_r = -\mu\left\{(-3r)(-a)(r^2+a^2)^{-5/2} + 3ra(r^2+a^2)^{-5/2}\right\}$$

$$= -6\mu a r(r^2+a^2)^{-5/2}$$

$$q_\theta = \frac{-\mu}{r}\left\{\cancel{-r(r^2+a^2)^{-3/2}} - 3ra(-a)(r^2+a^2)^{-5/2}\right.$$
$$\left.\cancel{+r(r^2+a^2)^{-3/2}} - 3ra(a)(r^2+a^2)^{-5/2}\right\}$$

$$= \frac{-\mu}{r}\left\{\cancel{3a^2(r^2+a^2)^{-5/2}} - \cancel{3a^2 r(r^2+a^2)^{-5/2}}\right\}$$

$$q_\theta = 0$$

$$q_\psi = 0$$

Along the stream line through $(r, \pi/2, \psi)$

Bernoulli's Equation,

$$\frac{P}{\rho} + \frac{1}{2}\bar{q}_r^{\,2} = \frac{P_\infty}{\rho}$$

$$\frac{P}{\rho} + \frac{1}{2}\left(-6\mu a r(r^2+a^2)^{-5/2}\right)^2 = \frac{P_\infty}{\rho}$$

$$\frac{P}{\rho} + \frac{1}{2}\,36\mu^2 a^2 r^2 (r^2+a^2)^{-5} = \frac{P_\infty}{\rho}$$

$$\Rightarrow P = P_\infty - 18\rho\mu^2 a^2 r^2 (r^2+a^2)^{-5}$$

$$\therefore \frac{dp}{dr} = -18\rho\mu^2 a^2\, 2r(r^2+a^2)^{-5} - 18\rho\mu^2 a^2 r^2(-5)(r^2+a^2)^{-6}(2r)$$

$$= -36\rho\mu^2 a^2 r(r^2+a^2)^{-5} + 36\times 5\rho\mu^2 a^2 r^3(r^2+a^2)^{-6}$$

$$= 36\rho\mu^2 a^2 r(r^2+a^2)^{-5}\left[-1 + \frac{5r^2}{r^2+a^2}\right]$$

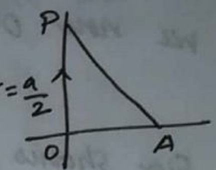
$$= 36\rho\mu^2 a^2 r(r^2+a^2)^{-5}\left(\frac{-r^2-a^2+5r^2}{r^2+a^2}\right)$$

$$\frac{dp}{dr} = 36\rho\mu^2 a^2 r(r^2+a^2)^{-6}(4r^2-a^2)$$

When $r = \frac{a}{2}$, $\frac{dP}{dr} = 0$ Also for $r = \frac{a+}{2}$, $\frac{dp}{dr} > 0$

and for $r = \frac{a-}{2}$, $\frac{dP}{dr} < 0$.

Hence p is a minimum at $r = \frac{1}{2}a$ on the plane



From the diagram

$$AP^2 = OP^2 + OA^2 = \frac{a^2}{4} + a^2 = \frac{5a^2}{4}$$

$$AP = \frac{a}{2}\sqrt{5}$$

$\therefore$ P is minimum at a distance $\frac{a\sqrt{5}}{2}$ from the doublet.

Axi-Symmetric Flow; Stoke's Stream Function

In a three dimensional motion if the flow of the fluid is the Same in all planes through a particular line then the motion is said to be axi-Symmetrical with this line as its axis.

Suppose the z- be taken an axis of Symmetry in an axi-symmetric flow and

Suppose that a point P in the fluid may be Specified by cylindrical polar coordinates (R, θ, z). Then at P all Scalar and vector functions associated with the flow are independent of θ.

The equation of continuity (for incompressible flow)

$$\nabla \cdot q = 0$$

$$\frac{\partial}{\partial R}(R q_R) + \frac{\partial}{\partial z}(R q_z) = 0 \longrightarrow (1)$$

taking $q = q_R \hat{R} + q_z \hat{k}$,

Find a Scalar function $\psi(R, z)$ Such that

$$\boxed{\begin{gathered} 2\pi\delta\psi = 2\pi R\,\delta z\, q_R - 2\pi R\,\delta R\, q_z \\ \frac{\partial\psi}{\partial z} = R q_R \;\&\; \frac{\partial\psi}{\partial R} = -R q_z \end{gathered}}$$

$$q_R = \frac{1}{R}\frac{\partial\psi}{\partial z} \quad \& \quad q_z = -\frac{1}{R}\frac{\partial\psi}{\partial R} \longrightarrow (2)$$

then on Sub. equ. (2) into (1),
we find the latter is identically Satisfied. Such a function ψ is called Stoke's stream function.
we now obtain a physical meaning for ψ.

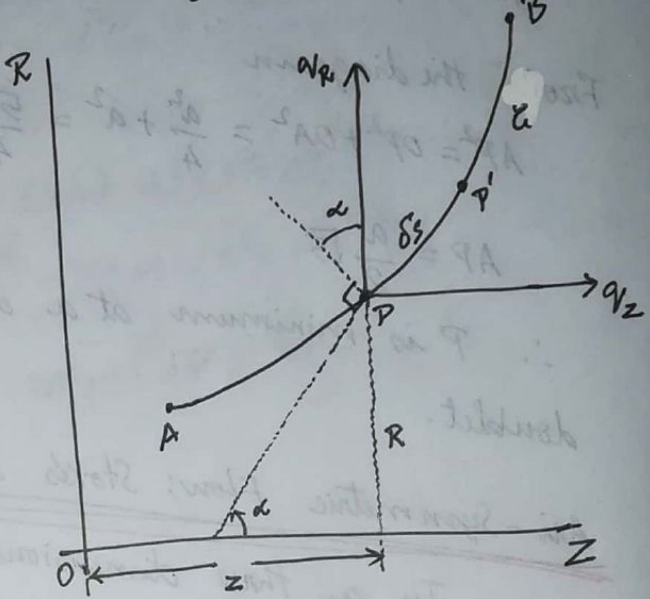


Fig. shows a meridian Section of the type of flow under discussion.

There is no velocity component perpendicular to the meridian plane.

In the Section, AB is an arc of a plane curve ξ, P being the point on it distant R from OZ and z from OR.

PP' is of length δs.

To find the volume of fluid crossing the Surface of revolution of AB about OZ per unit time from right to left.

Denoting by 'α' the angle which the tangent at P to ξ makes with OZ, the normal component of velocity at P from right to left is

$$q_R \cos\alpha - q_z \sin\alpha = \frac{1}{R}\frac{\partial \psi}{\partial z}\cos\alpha + \frac{1}{R}\frac{\partial \psi}{\partial R}\sin\alpha$$

$$= \frac{1}{R}\left(\cos\alpha \frac{\partial \psi}{\partial z} + \sin\alpha \frac{\partial \psi}{\partial R}\right)$$

$$= \frac{1}{R}\left(\frac{\partial z}{\partial s}\cdot\frac{\partial \psi}{\partial z} + \frac{\partial R}{\partial s}\cdot\frac{\partial \psi}{\partial R}\right)$$

$$= \frac{1}{R}\frac{\partial \psi}{\partial s}$$

Thus the volume of fluid crosses the Surface of revolution at PP' about OZ per unit time is

$$\frac{1}{R}\cdot\frac{\partial\psi}{\partial s}\times 2\pi R\,\delta s = 2\pi\,\delta s\,\frac{\partial\psi}{\partial s} = 2\pi\,\delta\psi \longrightarrow (I)$$

where $\delta\psi = \psi_{p}' - \psi_p$.

Hence the total volume of fluid crossing the surface of revolution of the arc AB about OZ per unit time from right to left is

$$\int_A^B 2\pi\,\delta\psi = 2\pi(\psi_B - \psi_A)$$

This quantity depends only on the position of A and B in the meridian section and not at all on the shape of ξ.

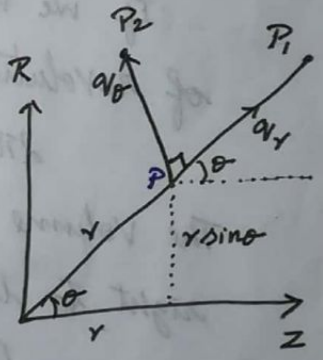
If ξ falls on the axis OZ its convenient to take $\psi_A = 0$,

so that the volume crossing the surface of relvolution of AP per unit time from right to left is $2\pi\psi$.

Further we note that as no fluid crosses a stream line,

ψ = constant along a stream

Also ψ = constant over a stream surface – the surface of revolution of a streamline about the axis of symmetry.



It is obtein useful to obtain velocity components in terms of spherical polar coordinates and derivatives of the stream functions

Fig. shows a merdian section through the axis of symmetry OZ of the flow, P being at distance r from O and at angular distance θ from $\overline{OZ}$.

The azimuthal coordinate is here redundant.

The fluid velocity components at P are q_r along $\overline{OP}$ and q_θ at right-angles to it in the sense of θ increasing.

The length PP_1 and PP_2 are respectively δr and $r\,d\theta$ (θ must not be confused with the angular coordinate used previously for cylindrical polar co-ordinates)

If we first take $\delta s = \delta r$, the area of the surface of the revolution of δs about OZ is $2\pi r \sin\theta\, \delta r$ and

the volume of fluid crossing this per unit time from right to left is $2\pi r \sin\theta\, \delta r\, q_\theta$.

Hence if ψ is the stream function at P, then

$$2\pi\, \delta\psi = 2\pi r \sin\theta\, \delta r\, q_\theta$$

$$\boxed{q_\theta = \frac{1}{r\sin\theta} \cdot \frac{\partial\psi}{\partial r}}$$

If we now take $\delta s = r\,d\theta$, the area of the surface of revolution about OZ is

$$2\pi r \sin\theta \cdot r\,\delta\theta = 2\pi r^2 \sin\theta\, \delta\theta.$$

The volume of fluid crossing this per unit time from right to left is

$(2\pi r^2 \sin\theta\, \delta\theta)(-q_r)$ and so

$$2\pi\, \delta\psi = (2\pi r^2 \sin\theta\, \delta\theta)(-q_r)$$

$$\boxed{q_r = -\frac{1}{r^2 \sin\theta} \frac{\partial\psi}{\partial\theta}}$$

—x—

Scanned with CamScanner

Some Special forms of the Stream function for Axis-Symmetric irrotational motion

The function ψ may be evaluated for a number of simple cases. The value of ψ for problems involving components of such standard distributions can then be obtained by superposition of the values of ψ for the separate components.

The Justification of superposing values of ψ depends on the fact that ψ satisfies a linear second-order partial differential equation, viz.

$$\frac{\partial^2 \psi}{\partial R^2} - \frac{1}{R}\frac{\partial \psi}{\partial R} + \frac{\partial^2 \psi}{\partial z^2} = 0 \longrightarrow (1)$$

in cylindrical polar coordinate (R, θ, z) when the flow is irrotational.

This equ. is derived from equations $q_R = \frac{1}{R}\frac{\partial \psi}{\partial z}$,

$q_z = -\frac{1}{R}\frac{\partial \psi}{\partial R}$

When the motion is irrotational $\nabla \times \bar{q} = 0$

$$\nabla \times \bar{q} = \begin{vmatrix} \vec{i} & \vec{j} & \bar{k} \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ \frac{1}{R}\frac{\partial \psi}{\partial z} & 0 & -\frac{1}{R}\frac{\partial \psi}{\partial R} \end{vmatrix}$$

$$= \vec{i}(0) - \vec{j}\left\{ \frac{\partial}{\partial R}\left(-\frac{1}{R}\frac{\partial \psi}{\partial R}\right) - \frac{\partial}{\partial z}\left(\frac{1}{R}\frac{\partial \psi}{\partial z}\right)\right\} + \bar{k}(0)$$

$$= -\vec{j}\left\{ \frac{\partial^2 \psi}{\partial R^2} - \underbrace{\cancel{\frac{1}{R}\frac{\partial \psi}{\partial R}}}_{0} + \frac{\partial^2 \psi}{\partial z^2}\right\} \quad \text{(by continuity equ.)}$$

$\nabla \times \bar{q} = 0$

$$\Rightarrow \frac{\partial^2 \psi}{\partial R^2} - \frac{1}{R}\frac{\partial \psi}{\partial R} + \frac{\partial^2 \psi}{\partial z^2} = 0 \longrightarrow (1)$$

Now using $q_z = -\frac{\partial \phi}{\partial z}$ $\quad q_R = -\frac{\partial \phi}{\partial R}$

$$\frac{\partial}{\partial R}(q_z) = \frac{\partial}{\partial R}\left(-\frac{\partial \phi}{\partial z}\right) = -\frac{\partial^2 \phi}{\partial R \partial z}$$

$$\frac{\partial}{\partial z}(q_r) = \frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial R}\right) = -\frac{\partial^2 \phi}{\partial R \partial z}$$

W.K.T
$$\frac{\partial^2 \phi}{\partial R \partial z} = \frac{\partial^2 \phi}{\partial z \partial R}$$

$$\therefore \frac{\partial}{\partial R}(q_z) = \frac{\partial}{\partial z}(q_R)$$

The Velocity Potential ϕ Satisfying the laplace equation in the co-ordinate system in the form

$$\frac{\partial^2 \phi}{\partial R^2} + \frac{1}{R}\frac{\partial \phi}{\partial R} + \frac{\partial^2 \phi}{\partial z^2} = 0 \longrightarrow (2)$$

Comparison of (1) & (2) Shows that ψ is not a harmonic function.

Uniform Stream:

Suppose we have a flow whose velocity is $-U\hat{k}$, where U is a constant. (i.e, uniform Stream U is parallel to $-ve$ direction of z axis)

Let P be the point in the Stream having cylindrical polar co-ordinates (R, θ, z) and ~~let~~

Let P_0 be its projection on the z-axis.

Taking ψ to the Stream function associated with P and zero that for P_0;

[Since the volume flowing from right to left through The circular disc obtained by revolving PP_0 about the z-axis is $\pi R^2 U$,]

$\therefore$ The uniform Stream Parallel $-ve$ direction of z-axis i.e., $q_z = -U$ and $q_R = 0$

But $q_z = -\frac{1}{R}\frac{\partial \psi}{\partial R}$ $\quad q_R = \frac{1}{R}\frac{\partial \psi}{\partial z}$

$$-\frac{1}{R}\frac{\partial \psi}{\partial R} = -U \Rightarrow \frac{\partial \psi}{\partial R} = UR$$

$\int$ing, $\psi = \frac{UR^2}{2} + f(z)$

Diff., w.r.t 'z',

$$\frac{\partial \psi}{\partial z} = f'(z)$$

$$\Rightarrow 0 = f'(z)$$

$$f(z) = \text{constant}$$

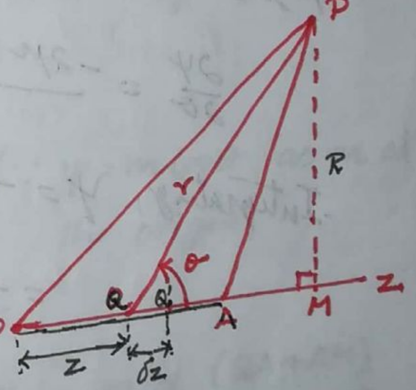
$$\therefore \psi = \frac{UR^2}{2} \quad (\text{omitting constant})$$

Interms of spherical polar coordinates

$$R = r\sin\theta$$

$$\therefore \psi = \frac{Ur^2\sin^2\theta}{2}$$

(ii) Simple Source.



Let ~~m~~ be the strength of the source at the origin.

W.K.T $q_r = \frac{m}{r^2}$, $q_\theta = 0$

using spherical polar coordinates

$$\therefore q_r = \frac{m}{r^2}, \quad q_\theta = 0$$

W.K.T $\quad q_r = \frac{-1}{r^2\sin\theta}\cdot\frac{\partial \psi}{\partial \theta}$

$$\frac{m}{\cancel{r^2}} = \frac{-1}{\cancel{r^2}\sin\theta}\cdot\frac{\partial \psi}{\partial \theta}$$

$$-m\sin\theta = \frac{\partial \psi}{\partial \theta}$$

$$\boxed{\psi = m\cos\theta + f(r)} \longrightarrow (1)$$

Diff. w.r.t 'r',

$$\frac{\partial \psi}{\partial r} = f'(r)$$

$$0 = f'(r)$$

$$f(r) = \text{constant}$$

$$(1) \Rightarrow \boxed{\psi = m\cos\theta}$$

iii) Doublet at O, Axis along $\overline{OZ}$

In Spherical polar coordinates the velocity potential at ϕ is $\phi(r,\theta) = \frac{\mu\cos\theta}{r^2}$

W.K.T $q_r = -\frac{\partial \phi}{\partial r} = -\mu\cos\theta \times \left(\frac{-2}{r^3}\right) = \frac{2\mu\cos\theta}{r^3}$

$$q_\theta = \frac{-1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{1}{r}\frac{\mu}{r^2}(-\sin\theta) = \frac{\mu}{r^3}\sin\theta$$

But $q_r = -\frac{1}{r^2\sin\theta}\frac{\partial \psi}{\partial \theta}$

$$\therefore \frac{-1}{r^2\sin\theta}\cdot\frac{\partial \psi}{\partial \theta} = \frac{2\mu\cos\theta}{r^3}$$

$$\frac{\partial \psi}{\partial \theta} = \frac{-2\mu r^2\sin\theta\cos\theta}{r^3}$$

Integrating, $\psi = -\frac{2\mu}{r}\int \sin\theta\cos\theta\, d\theta$

$$= -\frac{2\mu}{r}\int \frac{d(\sin^2\theta)}{2} + f(r)$$

$$\psi = -\frac{\mu\sin^2\theta}{r} + f(r) \longrightarrow (*)$$

Diff. w.r.t 'r', $\frac{\partial \psi}{\partial r} = \frac{\mu}{r^2}\sin^2\theta + f'(r) \longrightarrow (1)$

But $q_\theta = \frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r}$

$$\frac{\mu}{r^3}\sin^2\theta = \frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r}$$

$$\frac{\partial \psi}{\partial r} = \frac{\mu}{r^2}\sin^2\theta$$

$$(1) \Rightarrow \frac{M}{r^2}\sin^2\theta = \frac{M}{r^2}\sin^2\theta + f'(r)$$

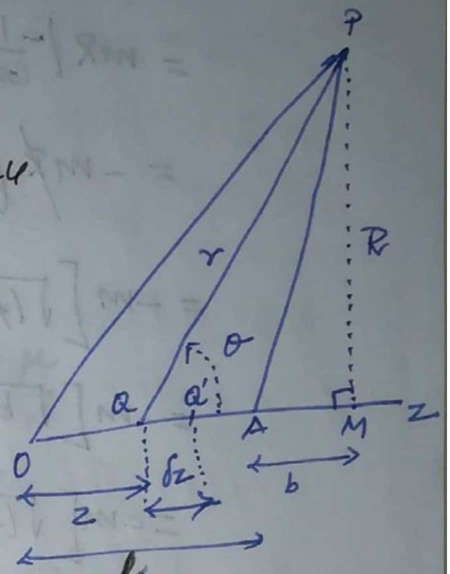
$$\Rightarrow f'(r) = 0$$

$$\Rightarrow f(r) = \text{Constant}$$

we get $\boxed{\psi = -\frac{M}{r}\sin^2\theta}$

(*) $\Rightarrow$

IV) Uniform line Source along $\overline{OZ}$



Let OA be a uniform line Source of Strength m per unit length along z-axis.

Let Q, Q' be neighbouring Points on OA, where $OQ = z$, $OQ' = z + \delta z$

Also Suppose $QP = r$, $\angle PQA = \theta$.

If m denotes the strength per unit length of the line Source the QQ' is effectively a Simple Source at Q of Strength $m\delta z$.

Hence the contribution to ψ at P from this Source is $\delta\psi$,

where $\delta\psi = m\delta z\cos\theta$ $(\because \psi = m\cos\theta$ as m at the origin)

Denoting $OA = l$, $AM = b$, $PM = R$ all of which are constant elements.

$$\delta\psi = m\delta z\cos\theta = m\delta z\frac{QM}{PQ} = \frac{m\delta z\,(QA + AM)}{\sqrt{QM^2 + PM^2}}$$

$$= \frac{m\delta z\,(OA - OQ + AM)}{\sqrt{(OM - OQ)^2 + PM^2}} = \frac{m\delta z\,(l + b - z)}{\sqrt{(l+b-z)^2 + R^2}}$$

Let $l + b - z = R\tan\theta$, $-dz = -R\sec^2\theta\, d\theta$

$$\int d\psi = m\int_0^l \frac{(l+b-z)\,dz}{\{(l+b-z)^2 + R^2\}^{1/2}} = m\int_0^l \frac{R\tan\theta\,(-R\sec^2\theta)\,d\theta}{(R^2\tan^2\theta + R^2)^{1/2}}$$

$$= -m\int_0^l \frac{R^{\cancel{2}}\tan\theta\,\sec^{\cancel{2}}\theta}{\cancel{R}\cancel{\sec\theta}}\,d\theta = -mR\int_0^l \tan\theta\,\sec\theta\,d\theta$$

$$= -mR \int_0^{l} \frac{\sin\theta}{\cos^2\theta}\, d\theta$$

Put $t = \cos\theta$, $dt = -\sin\theta\, d\theta$

$$\psi = -mR \int_0^{l} \frac{-dt}{t^2} = mR\left(\frac{t^{-2+1}}{-2+1}\right)_0^{l} = mR\left(-\frac{1}{t}\right)_0^{l}$$

$$= mR\left(-\frac{1}{\cos\theta}\right)_0^{l}$$

$$= -m\cancel{R}\left(\frac{\sqrt{(l+b-z)^2+R^2}}{\cancel{R}}\right)_0^{l}$$

$$= -m\left[\sqrt{(\cancel{l}+b-\cancel{l})^2+R^2} - \sqrt{(l+b-0)^2+R^2}\right]$$

$$= -m\left[\sqrt{b^2+R^2} - \sqrt{(l+b)^2+R^2}\right]$$

$$= m\left[\sqrt{(l+b)^2+R^2} - \sqrt{b^2+R^2}\right]$$

$$= m\left[\sqrt{OM^2+PM^2} - \sqrt{AM^2+PM^2}\right]$$

$$= m\left[\sqrt{OP^2} - \sqrt{AP^2}\right]$$

$$= m(OP - AP)$$

The stream lines are given by

$$\psi = \text{constant}$$

$$\psi = m(OP - AP) = \text{constant}$$

$$OA - AP = \text{constant}$$

These are confocal hyperboloids of revolution about OZ, having O and A as foci.

v) Doublet in Uniform Stream

Let there be a uniform stream U parallel to the $-ve$ z-axis.

Let there be a doublet of strength μ kept at the origin with its axis parallel to the z-axis.

Scanned with CamScanner

Then the Stream function is $\psi = \frac{U}{2} r^2 \sin^2\theta - \frac{\mu}{r}\sin^2\theta$

The stream line are given by ψ = constant

i.e, $\frac{U}{2} r^2 \sin^2\theta - \frac{\mu}{r}\sin^2\theta = \text{constant}$

In particular the surface for which $\psi = 0$ are given by

$$\frac{1}{2} U r^2 \sin^2\theta - \frac{\mu}{r}\sin^2\theta = 0$$

$$\sin^2\theta\left[\frac{Ur^2}{2} - \frac{\mu}{r}\right] = 0$$

$$\frac{Ur^2}{2} - \frac{\mu}{r} = 0 \Rightarrow \frac{Ur^2}{2} = \frac{\mu}{r}$$

$$r^3 = \frac{2\mu}{U}$$

$\sin^2\theta = 0$ $\qquad\qquad$ $r = \left(\frac{2\mu}{U}\right)^{1/3}$

$\sin\theta = 0$

$\theta = \sin^{-1}(0)$

$\theta = 0, \pi$

$\theta = 0, \pi$ is the z-axis and $r = \left(\frac{2\mu}{U}\right)^{1/3}$ represent a sphere with centre at the origin and radius $\left(\frac{2\mu}{U}\right)^{1/3}$

---

1) Define the Stokes Stream function $\psi(r, \theta)$ (in Spherical polar coordinates (r, θ, ϕ) for the axi-symmetric flow of an incompressible fluid. Determine the stream function corresponding

(i) to a uniform stream U parallel to the axis $\theta = 0$,

and (ii) to the spherically symmetric radial velocity field from a point source at the origin, the total outward flux being $4\pi m$.

The equation $r\sin\theta = 2a\cos\frac{1}{2}\theta$ represents the surface of a rigid blunt-nosed cylinder, symmetric about the axis $\theta = 0$. Inviscid fluid flows irrotationally past this cylinder, the velocity far from the ~~pressure distribution, as a~~ cylinder being U parallel to the cylinder axis. Show that the Sum of the two Stream functions (i) and (ii) above, with $m = a^2 U$, may be used to represent this flow, and find the pressure distribution, as a function of θ, on the cylinder surface.

Sol. W.K.T uniform Stream U Parallel to -ve direction of Z-axis the Stream function

$$\psi = \frac{U}{2} r^2 \sin^2\theta$$

Now determine the stream functions corresponding to a uniform stream.

i.e) The direction of the -ve Z-axis.

In that case $\psi = -\frac{U}{2} r^2 \sin^2\theta$

(iii) The stream function due to the Simple Source of strength $4\pi M$ at the orgin can also be obtained from the standard result, but it is instructive to find it by an alternative process.

Let P be the point (r, θ, ϕ) and P_0 the point on $\theta = 0$ distance r from O. When the circular arc centre O and radius $OP = OP_0 = r$ is revolved once round the line $\theta = 0$, it generates a cap of a Sphere whose area is equal to its projection on the corresponding enveloping cylinder with axis along $\theta = 0$.

Since the length of the projection is $r(1 - \cos\theta)$ the area of the cap is $2\pi r^2 (1 - \cos\theta)$. ($\because$ AB = projection of AC = AC $\cos\theta$)

In unit time $4\pi m$ units of Volume of fluid

Scanned with CamScanner

emerges from O.

Hence the volume flux per unit time across the cap is

$$\frac{2\pi r^2(1-\cos\theta)}{4\pi r^2}\times 4\pi m = 2m(1-\cos\theta)$$

from left to right. [$\because$ volume of the fluid across the sphere per unit is $4\pi r^2 q_r = 4\pi m \Rightarrow q_r = m/r^2$]

Thus $-2m(1-\cos\theta)$ is the flux from right to left and as this is $2\pi\psi$, we have

$$\psi = -m(1-\cos\theta), \text{ for the source}$$

Thus the total stream function is

$$\psi = -m(1-\cos\theta) - \frac{1}{2}Ur^2\sin^2\theta$$

Taking $m = a^2U$, $\psi = -(a^2U)(1-\cos\theta) - \frac{1}{2}Ur^2\sin^2\theta$

$$\psi = -U\{a^2(1-\cos\theta) + \frac{1}{2}r^2\sin^2\theta\} \longrightarrow (1)$$

The stream surfaces are given by

$\psi = \text{const.} \Rightarrow -U\{a^2(1-\cos\theta) + \frac{1}{2}r^2\sin^2\theta\} = \text{Constant}$

$\cancel{2a^2\cos^2\frac{1}{2}\theta - \frac{1}{2}r^2\sin^2\theta = \text{constant}}$

~~The surface or surfaces for which~~

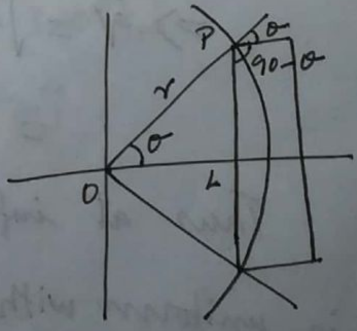
$$-\{a^2(1-\cos\theta) + \frac{r^2\sin^2\theta}{2}\} = \text{constant}$$

$$-a^2 + a^2\cos\theta - \frac{r^2\sin^2\theta}{2} = \text{constant}$$

$$-a^2 + a^2(2\cos^2\theta/2 - 1) - r^2/2\,\sin^2\theta = \text{const.}$$

$$\Rightarrow 2a^2\cos^2\theta/2 - 2a^2 - r^2/2\,\sin^2\theta = \text{const.} \longrightarrow (2)$$

The surface for which the constant on the R.H.S of (2) is zero are given by



$$2a^2\cos^2\theta/2 - \frac{1}{2}r^2\sin^2\theta = 0$$

$$r^2\sin^2\theta = 4a^2\cos^2\theta/2$$

$$\Rightarrow r\sin\theta = 2a\cos\theta/2$$

$$r = \frac{2a\cos\theta/2}{\sin\theta} = \frac{\cancel{2}a\cancel{\cos\theta/2}}{\cancel{2}\sin\theta/2\cancel{\cos\theta/2}} = a\,\mathrm{cosec}\,\theta/2 \longrightarrow (3)$$

Eqn. (3) is Satisfied by

$\cos\theta/2 = 0 \Rightarrow \frac{\theta}{2} = \cos^{-1}(0) = \frac{\pi}{2}$

$\theta = \pi$

When $r = a\,\mathrm{cosec}\,\theta/2$

$q_r = 0, \quad q_r = \frac{-1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}$

Thus the given cylinder can be fited to the Stream Surface (3)

The velocity components q_r, q_θ at any point of the fluid are found from (1)

Hence
$$q_r = \frac{-1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}$$

$$= \frac{-1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left\{-U\left[a^2(1-\cos\theta)+\frac{1}{2}r^2\sin^2\theta\right]\right\}$$

$$= \frac{U}{r^2\sin\theta}\left[a^2\sin\theta + \frac{r^2}{\cancel{2}}\cancel{2}\sin\theta\cos\theta\right]$$

$$= \frac{U}{\cancel{r^2}\cancel{\sin\theta}}\cancel{r^2}\cancel{\sin\theta}\left[\frac{a^2}{r^2}+\cos\theta\right]$$

$$q_r = U\left[\frac{a^2}{r^2}+\cos\theta\right]$$

$$q_\theta = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} = \frac{1}{r\sin\theta}\frac{\partial}{\partial r}\left\{-U\left[a^2(1-\cos\theta) + \frac{1}{2}r^2\sin^2\theta\right]\right\}$$

$$= \frac{-U}{\cancel{r}\cancel{\sin\theta}}\left[0 + \frac{1}{\cancel{2}}\cancel{2}\cancel{r}\sin^{\cancel{2}}\theta\right]$$

$$q_\theta = -U\sin\theta$$

As $r \to \infty$, $q_r = U\cos\theta$, $q_\theta = -U\sin\theta$

$$\Rightarrow \vec{q} = \sqrt{q_r^2+q_\theta^2} = \sqrt{U^2\cos^2\theta + U^2\sin^2\theta}$$

$$= \sqrt{U^2(\sin^2\theta+\cos^2\theta)} = \sqrt{U^2} = U$$

Thus at infinite distance from $\theta = 0$ the flow is uniform with velocity U Parallel to $\theta = 0$

Thus (1) is the appropriate stream function for flow past the given cylinder

The fluid Speed $\bar{q}$ at any point is

$$\bar{q} = \sqrt{q_r^2 + q_\theta^2} = \sqrt{U^2\left(\frac{a^2}{r^2} + \cos^2\theta + \frac{2a^2}{r^2}\cos\theta + \sin^2\theta\right.}$$

$$= U\sqrt{1 + \frac{2a^2}{r^2}\cos\theta + \frac{a^4}{r^4}}$$

On the cylinder Surface $r = a\,\mathrm{cosec}\,\theta/2$, we obtain

$$\bar{q} = U\sqrt{1 + \frac{2\cancel{a^2}\cos\theta}{\cancel{a^2}\,\mathrm{cosec}^2\theta/2} + \frac{\cancel{a^4}}{\cancel{a^4}\,\mathrm{cosec}^4\theta/2}}$$

$$= U\sqrt{1 + 2\cos\theta\,(\sin^2\theta/2) + \sin^4\theta/2}$$

$$= U\sqrt{1 + \cancel{2}\cos\theta\left(\frac{1-\cos\theta}{\cancel{2}}\right) + \left(\frac{1-\cos\theta}{2}\right)^2}$$

$$= U\sqrt{1 + \cos\theta(1-\cos\theta) + \left(\frac{1+\cos^2\theta - 2\cos\theta}{4}\right)}$$

$$= \frac{U}{2}\sqrt{4 + 4\cos\theta - 4\cos^2\theta + 1 + \cos^2\theta - 2\cos\theta}$$

$$= \frac{U}{2}\sqrt{5 + 2\cos\theta - 3\cos^2\theta}$$

Now apply Bernoulli's Equ. along the entire Surface $\psi = 0$

Taking $p = P_\infty$ at $r = \infty$ on this Surface

$$\frac{P}{\rho} + \frac{1}{2}q^2 = \frac{P_\infty}{\rho} + \frac{1}{2}U^2$$

$$\Rightarrow P = P_\infty + \frac{\rho}{2}U^2 - \frac{1}{2}\rho q^2$$

$$= P_\infty + \frac{\rho}{2}U^2 - \frac{\rho}{2}\frac{U^2}{4}(5 + 2\cos\theta - 3\cos^2\theta)$$

$$= P_\infty - \frac{U^2\rho}{8}[-4 + 5 + 2\cos\theta - 3\cos^2\theta]$$

$$P = P_\infty - \frac{U^2\rho}{8}[1 + 2\cos\theta - 3\cos^2\theta]$$

—x—